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Splitting the shadow

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Abstract

We derive formulae for the theta series of the two translates of the even sublattice L_0 of an odd unimodular lattice L that constitute the shadow of L. The proof rests on special evaluations of the Jacobi theta series attached to L and to a certain vector. We produce an analogous theorem for codes. Additionally, we construct non-linear formally self-dual codes and relate them to lattices.

Keywords: Jacobi forms; Unimodular lattices; Self-dual codes; Shadows

1. Introduction

The shadow of a lattice has received some attention since the landmark paper [6], where it was employed to derive upper bounds on the minimum norm of unimodular odd lattices. The shadow of a code was described in [5] and numerous papers have generalized these results. In [5], a careful study of congruence properties of norms of vectors led to extension constructions for unimodular lattices and self-dual codes. Building on these latter results, in the present note we derive closed formulae for the theta series of the two translates of the even sublattice L_0 of an odd unimodular lattice L, that constitute the shadow of L. These formulae can be made more explicit in the case of a lattice obtained via Construction A_{2k} from a code over $\mathbf{Z}_{2k}$. In a similar

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manner we derive an analogous theorem for self-dual codes over $\mathbf{Z}_{2k}$. An important tool is the Jacobi theta series introduced in [10] and studied further in [4].

2. Definitions and notations

2.1. Lattices

An n-dimensional lattice is a discrete additive subgroup of $\mathbf{R}^n$. We attach the standard inner-product, i.e. for vectors x and y

$$x \cdot y = \sum x_i y_i.$$

The norm of x in $\mathbf{R}^n$ is $x \cdot x$. The dual L^* of a lattice L is defined as

$$L^* := \{y \in \mathbf{R}^n \mid \forall x \in L, x \cdot y \in \mathbf{Z}\}.$$

A lattice is *unimodular* if it is equal to its dual. A unimodular lattice is Type II if all its vectors have even norms, Type I otherwise. Consider a Type I lattice L. Let L_0 denote the sublattice of even norm vectors of L and L_2 its unique non-trivial coset in L. Call further L_1 and L_3 the other two cosets of L_0 in L_0^*. The unique non-trivial coset of L in L_0^* is called the *shadow* of L (denoted by S) and is equal to $L_1 \cup L_3$.

2.2. Theta series

The ordinary theta series of a lattice L is

$$\theta_L(\tau) := \sum_{x \in L} q^{x \cdot x},$$

where $q = \exp(i\pi\tau)$, with $\tau \in \mathbf{C}$ and $\Im(\tau) > 0$.

The *Jacobi theta series* attached to a lattice L and a vector $y \in \mathbf{R}^n$ is

$$\theta_{L,y}(\tau, z) := \sum_{x \in L} q^{x \cdot x} \xi^{y \cdot x},$$

where q is as before and $\xi = \exp(2\pi i z)$, with $z \in \mathbf{C}$. For each k and $i = 0, 1, 2, \ldots, 2k-1$ put

$$t_i(\tau, z) = \sum_{r \equiv i \,(\mathrm{mod}\, 2k)} q^{r^2/2k} \xi^r,$$

where q and ξ are as before and let $T_i(\tau) = t_i(\tau, 0)$. Further, for any real a let

$$t_{i,a}(\tau, z) := t_i(\tau, az).$$

2.3. $\mathbf{Z}_{2k}$-codes

A linear code over $\mathbf{Z}_{2k}$ is a submodule of $\mathbf{Z}_{2k}^n$. We attach the standard inner product to the space, that is $[v, w] = \sum v_i w_i$. The dual $C^\perp$ is understood with respect to this inner product. A code is *self-dual* if it is equal to its dual. The Euclidean weight of a vector $x=(x_1,x_2,\dots,x_n)$ is $\sum_{i=1}^n \min\{x_i^2,(2k-x_i)^2\}$. A code is Type II if all vectors in the code have Euclidean weights which are $0 \pmod{4k}$ and Type I otherwise. If C is a Type I code over Z_{2k} and C_0 is the subcode of vectors whose Euclidean weight is $0 \pmod{4k}$ then $C_2 = C - C_0$ and the shadow is $C_0^{\perp} - C = C_1 \cup C_3$, see [1] for a complete description.

We shall recall the standard A_{2k} construction of a lattice from a self-dual code over $\mathbf{Z}_{2k}$. Define the reduction modulo $2k$, by $\rho : \mathbf{Z}^n \to \mathbf{Z}_{2k}^n$, by

$$\rho(x_1,\dots,x_n) = (x_1 \pmod{2k}, \dots, x_n \pmod{2k}).$$

Given a code C over $\mathbf{Z}_{2k}$ we construct a lattice by

$$\Lambda(C) = \frac{1}{\sqrt{2k}} \{x \in \mathbf{Z}^n \mid \rho(x) \in \mathbf{C}\}. \tag{1}$$

It is shown in [1] that if C is a Type I code then $\Lambda(C)$ is a Type I unimodular lattice, and that if C is a Type II code then $\Lambda(C)$ is a Type II unimodular lattice and that the minimum norm of the lattice is $\min\{2k, d_{\mathrm{E}}/2k\}$, where d_{E} is the minimum Euclidean weight of the code. Moreover, it is shown that the image of the shadow under Λ is the shadow of the image, see [8] for a complete explanation of the connection between shadow codes and shadow lattices.

A special code we shall use later is the even code E_n over $\mathbf{Z}_4$ which is defined as $E_n := 2\mathbf{Z}_4^n$. Its complete weight enumerator (defined below) is

$$\mathrm{cwe}_{E_n} = (x_0 + x_2)^n.$$

2.4. Weight enumerators

Define the complete weight enumerator for a code C over $\mathbf{Z}_{2k}$ by

$$\mathrm{cwe}_C(x_0,x_1,\dots,x_{2k-1}) = \sum A_{a_0,a_1,\dots,a_{2k-1}} x_0^{a_0} x_1^{a_1} \dots x_{2k-1}^{a_{2k-1}}, \tag{2}$$

where there are $A_{a_0,a_1,\dots,a_{2k-1}}$ vectors with a_i coordinates with an i. The symmetric weight enumerator is

$$\mathrm{swe}_C(x_0,x_1,\dots,x_{2k-1}) = \sum A_{a_0,a_1,\dots,a_k} x_0^{a_0} x_1^{a_1} \dots x_k^{a_k}, \tag{3}$$

where there are $A_{a_0,a_1,\dots,a_k}$ vectors with a_i coordinates with an $\pm i$. The Hamming weight enumerator is given by $H_C(x,y) = \mathrm{swe}(x,y,y,\dots,y)$. The minimum Euclidean and Hamming weights of a code are denoted by d_{E} and d_{H}. The Lee weight of a vector over $\mathbf{Z}_4$ is the sum of the Lee weights of each component. The elements have Lee weight corresponding to their binary image under the gray map, specifically, $0,1,2,3$ have Lee weight $0,1,2$, and 1, respectively. The minimum Lee weight of a $\mathbf{Z}_4$ code is denoted d_{Lee}.

We introduce the following weight enumerator. For a code C and a vector y define

$$J_{C,y} = \sum_{c\in C} x_{i,j}^{n_{ij}(c)}, \tag{4}$$

where $n_{ij}(c)$ is the number of coordinates that have an i in c and a j in y.

Observe that for $c \in C$,

$$c \cdot y = \sum_{i,j} n_{ij}(c)ij.$$

3. Evaluations

3.1. Lattices

We shall state the main result of this section and then give the necessary lemmas to prove this theorem. The main result of this section is the following.

Theorem 1. *Let L be an odd unimodular lattice of dimension n. Let L_0 denote the sublattice of even norm vectors with L_2 the unique non-trivial coset in L, and let L_1 and L_3 be the other two cosets in L_0^* with the shadow $S = L_1 \cup L_3$. Set*

$$\mu_n(\tau) = \exp\left(\frac{\mathrm{i}\pi n}{2}\left(1 - \frac{1}{\tau}\right)\right).$$

Let y denote an arbitrary element of L_1. Then if $n \equiv 0 \pmod 2$ then the theta series Θ_1 and Θ_3 of L_1 and L_3 evaluate as

$$2\Theta_1(\tau) = \left(\frac{\mathrm{i}}{\tau}\right)^{n/2}\left(\theta_L\left(1 - \frac{1}{\tau}\right) + \mu_n(\tau)\theta_{L,y}\left(1 - \frac{1}{\tau}, \frac{1}{\tau}\right)\right),$$

$$2\Theta_3(\tau) = \left(\frac{\mathrm{i}}{\tau}\right)^{n/2}\left(\theta_L\left(1 - \frac{1}{\tau}\right) - \mu_n(\tau)\theta_{L,y}\left(1 - \frac{1}{\tau}, \frac{1}{\tau}\right)\right).$$

If $n \equiv 1 \pmod 2$ then

$$\Theta_1(\tau) = \Theta_3(\tau) = \tfrac{1}{2}\theta_S(\tau).$$

We prepare for the proof by a pair of lemmata. First we note the immediate.

Lemma 1. $\Theta_1(\tau) + \Theta_3(\tau) = (\mathrm{i}/\tau)^{n/2}\theta_L(1 - 1/\tau)$.

Proof. We express $\theta_S(\tau)$ in two ways by $S = L_1 \cup L_3$ and by Conway and Sloane [7, (4) p. 440], that is

$$\theta_{L_0^*}(\tau) - \theta_L(\tau) = \left(\frac{\mathrm{i}}{\tau}\right)^{n/2}\theta_L\left(1 - \frac{1}{\tau}\right). \qquad \square \tag{5}$$

We proceed by generalizing [7, (4) p. 440] from the theta series to the Jacobi theta series. That is, we express the Jacobi theta series of the shadow as a function of the Jacobi theta series of the lattice.

Lemma 2. *For a Type I unimodular lattice L and any vector $y \in \mathbf{R}^n$ we have*

$$\theta_{S,y}(\tau,z) = \left(\frac{\mathrm{i}}{\tau}\right)^{n/2} \exp\left(-\mathrm{i}\pi\,\frac{z^2(y\cdot y)}{\tau}\right)\theta_{L,y}\left(1-\frac{1}{\tau},\frac{z}{\tau}\right).$$

Proof. First we express $\theta_{L_0,y}$ as a function of $\theta_{L,y}$.

$$\theta_{L_0,y}(\tau,z) = \tfrac{1}{2}(\theta_{L,y}(\tau,z) + \theta_{L,y}(\tau+1,z)).$$

Then we use the Poisson Jacobi formula [4,10] to express $\theta_{L_0,y}$ as a function of $\theta_{L_0^*,y}$ and $\theta_{L,y}$ as a function of $\theta_{L^*,y}$. The result follows. $\square$

We can now sketch a proof of Theorem 1.

Proof. We compute $\Theta_1 - \Theta_3$ by splitting the range of summation in the defining equation of $\theta_{S,y}(\tau,1)$ and using the tables for $n \equiv 0 \pmod 2$ in [8] which give the orthogonality relations between the cosets L_i, to observe that the power of ξ is a constant for $x \in L_i$ and $y \in L_1$. The value of $\theta_{S,y}(\tau,1)$ can then be obtained from Lemma 2.

Since by Lemma 1 we know $\Theta_1 + \Theta_3$ we conclude by solving a system of two equations in two unknowns, Θ_1 and Θ_2.

For the cases when $n \equiv 1 \pmod 2$ we have that the glue group is the cyclic group of order 4, and that $L_1 = -L_3$. It follows that these theta series are equal. $\square$

3.2. Codes

Throughout this section let C be a Type I code and C_0 its subcode of doubly even vectors, and $C_2 = C - C_0$ with $S = C_0^{\perp} - C = C_1 \cup C_3$. Let ζ_g denote a g-th root of unity. The matrix $A = (a_{ij})$ is a $2k \times 2k$ matrix with

$$a_{ij} = \frac{1}{\sqrt{2k}}\,\zeta_{4k}^{i^2+ij}.$$

We shall now give an analog to Theorem 1 for codes over $\mathbf{Z}_{2k}$.

Theorem 2. *Let C be a Type I code of length n. Let C_0 denote the subcode of even vectors with C_2 the unique non-trivial coset in C, and let C_1 and C_3 be the other two cosets in $C_0^{\perp}$ with shadow $C_1 \cup C_3$. Let y denote a constant vector of C_1. Then if $n \equiv 0 \pmod 2$ then the complete weight enumerators of C_1 and C_3 evaluate as*

$$2\mathrm{cwe}_{C_1}(x_0,x_1,\ldots,x_{2k-1}) = \mathrm{cwe}_C(A(x_0,x_1,\ldots,x_{2k-1})) + (-1)^{n/2} J_{S,y}(\zeta_{2k}^{ij}x_{i\cdot j}), \quad (6)$$

$$2\mathrm{cwe}_{C_3}(x_0,x_1,\ldots,x_{2k-1}) = \mathrm{cwe}_C(A(x_0,x_1,\ldots,x_{2k-1})) - (-1)^{n/2} J_{S,y}(\zeta_{2k}^{ij}x_{i\cdot j}). \quad (7)$$

If $n \equiv 1 \pmod 2$ then

$$\mathrm{swe}_{C_1}(x_0,\ldots,x_k) = \mathrm{swe}_{C_3}(x_0,\ldots,x_k) = \tfrac{1}{2}\mathrm{swe}_S(x_0,\ldots,x_k).$$

We have the following analog to Lemma 1.

Lemma 3. *Let C be a Type I code and A the matrix as defined above, then*

$$\mathrm{cwe}_C(A(x_0,x_1,\ldots,x_{2k-1})) = \mathrm{cwe}_{C_1}(x_0,x_1,\ldots,x_{2k-1}) + \mathrm{cwe}_{C_3}(x_0,x_1,\ldots,x_{2k-1}).$$

Proof. We express $\mathrm{cwe}_S(x_0,x_1,\ldots,x_{2k-1})$ in two ways by $S=C_1\cup C_3$ and by Bannai et al. [1, Theorem 6.2, p. 1201], that is

$$\mathrm{cwe}_S(x_0,x_1,\ldots,x_{2k-1}) = \mathrm{cwe}_C(A(x_0,x_1,\ldots,x_{2k-1})). \qquad \square \tag{8}$$

Consider the polynomial $J_{C,y}=\sum_{c\in C} x_{ij}^{n_{ij}(c)}$. We note that for $c\in C$, $c\cdot y=\sum_{i,j} n_{ij}(c)ij$, and that this product is constant for $c\in C_0$, $y\in C_1$ and $c\in C_0$, $y\in C_1$. Hence, it is most useful when $y\in S$, the shadow of the code.

From [1] (corrected in [3]) we have

$$J_{S,y}(X_{ij}) = \frac{1}{|C|}(T\otimes I)\cdot J_{C,y}(X_{\phi(\mathbf{a})}), \tag{9}$$

where $T_{a,b}=(\zeta_{4k})^{ab}$ with $a,b\in\mathbf{Z}_{2k}$ and $\phi(\mathbf{a})=\zeta_{4k}^{b^2}(a,b)$ with $\mathbf{a}=(a,b)$.

Let $y\in S$ and substitute $X_{ij}=z^{ij}x_{i\cdot j}$ in $J_{S,y}(X_{ij})$. Splitting the range of summation we have

$$J_{S,y}(z_{ij}^{ij}x_{i\cdot j}) = z^{c_1\cdot y}\mathrm{cwe}_{C_1}(x_{i\cdot j}) + z^{c_3\cdot y}\mathrm{cwe}_{C_3}(x_{i\cdot j}), \tag{10}$$

where $c_i\cdot y$ represents the constant inner product of y with an element of C_i. Note that it was imperative that y be a constant vector for Eq. (10) to hold.

Using the tables in [8] which give the orthogonality relations between the cosets C_i, we get the following lemma.

Lemma 4. *Let C be a Type I code then,*

$$\mathrm{cwe}_{C_1}(x_{i\cdot j}) - \mathrm{cwe}_{C_3}(x_{i\cdot j}) = (-1)^{n/2}J_{S,y}(\zeta_{2k}^{ij}x_{i\cdot j}). \tag{11}$$

The proof of Theorem 2 follows directly from the previous lemmata and that fact that when n is odd, $C_1=-C_3$.

We give an elementary example of Theorem 2. Let $C=E_2=\{(00),(22),(20),(02)\}$. Then $C_0=\{(00),(22)\}$, $C_2=\{(02),(20)\}$, $C_1=\{(11),(33)\}$, and $C_3=\{(13),(31)\}$. Then $W_{C_1}=x_1^2+x_3^2$ and $W_{C_3}=2x_1x_3$. Choose $y=(11)$. We have $J_{S,y}(X_{ij})=x_{11}^2+x_{13}^2+2x_{11}x_{13}$, then $J_{S,y}(\sqrt{-1}x_{i\cdot j})=-x_1^2-x_3^2+2x_1x_3$. Finally,

$$\begin{aligned} W_C(A(x_0,x_1,x_2,x_3)) &- J_{S,y}(\sqrt{-1}x_{i\cdot j}) \\ &= x_1^2+x_3^2+2x_1x_3+x_1^2+x_3^2-2x_1x_3 \\ &= 2x_1^2+2x_3^2 = 2W_{C_1} \end{aligned}$$

and

$$\begin{aligned} W_C(A(x_0,x_1,x_2,x_3)) &- J_{S,y}(\sqrt{-1}x_{i\cdot j}) \\ &= x_1^2+x_3^2+2x_1x_3-x_1^2-x_3^2+2x_1x_3 \\ &= 4x_1x_3 = 2W_{C_3}. \end{aligned}$$

Note that if vector (13) is used then the theorem does not hold since it is not a constant vector.

4. Applications

4.1. Construction A_{2k}

Theorem 1 can only be useful if we know how to compute $\theta_{S,y}$. Following [7] we shall denote by $[a]$ the vector

$$[a]=(a/2,\ldots,a/2).$$

We shall require the following result from [4].

Lemma 5 (Choie and Kim [4]). *If L is a Type I lattice obtained by Construction A_{2k} from a code C then*

$$\theta_{L,[a]}(\tau,z)=\mathrm{cwe}_C(t_{0,a}(\tau,z),t_{1,a}(\tau,z),t_{2,a}(\tau,z),\ldots,t_{2k-1,a}(\tau,z)).$$

Combining this lemma with Theorem 1 we obtain

Theorem 3. *With the notations of Theorem* 1 *we have for a Type I lattice, whose shadow contains* [a], *the following identities hold*:

$$\begin{aligned}2\Theta_1(\tau)=\left(\frac{\mathrm{i}}{\tau}\right)^{n/2}&\left(\mathrm{cwe}_C\left(T_0\left(1-\frac{1}{\tau}\right),T_1\left(1-\frac{1}{\tau}\right),\right.\right.\\&T_2\left(1-\frac{1}{\tau}\right),\ldots,T_{2k-1}\left(1-\frac{1}{\tau}\right)\bigg)+\mu_n\,\mathrm{cwe}_C\left(t_{0,a}\left(1-\frac{1}{\tau},\frac{1}{\tau}\right),\right.\\&\left.\left.t_{1,a}\left(1-\frac{1}{\tau},\frac{1}{\tau}\right),\ldots,t_{2k-1,a}\left(1-\frac{1}{\tau},\frac{1}{\tau}\right)\right)\right),\\2\Theta_3(\tau)=&\left(\mathrm{cwe}_C\left(T_0\left(1-\frac{1}{\tau}\right),T_1\left(1-\frac{1}{\tau}\right),T_2\left(1-\frac{1}{\tau}\right),\ldots,T_{2k-1}\left(1-\frac{1}{\tau}\right)\right)\right.\\&-\mu_n\,\mathrm{cwe}_C\left(t_{0,a}\left(1-\frac{1}{\tau},\frac{1}{\tau}\right),t_{1,a}\left(1-\frac{1}{\tau},\frac{1}{\tau}\right),\ldots,\right.\\&\left.\left.t_{2k-1,a}\left(1-\frac{1}{\tau},\frac{1}{\tau}\right)\right)\right).\end{aligned}$$

4.2. Shadow sums and extensions

The following construction while implicit in [7] was first defined in [8]. It generalizes the extensions of [8].

Theorem 4 (Dougherty and Solé [9]). *Let L and L' denote two Type I unimodular lattices of respective dimensions n and n'. The set*

$$L \oplus_S L' := \bigcup_{i=0}^{3} L_i \times L_i'$$

is a unimodular lattice of dimension $n+n'$. It is Type II if $n+n'$ is a multiple of 8. *Let C and C' denote two Type I self-dual codes over $\mathbf{Z}_{2k}$ of respective lengths n and n'. The set*

$$C \oplus_S C' := \bigcup_{i=0}^{3} C_i \times C_i'$$

is a self-dual code of length $n+n'$. It is Type II if $n+n'$ is a multiple of 8.

For instance:

- $\mathbf{Z}^i \oplus_S \mathbf{Z}^{8-i} = E_8$ for $0 < i < 8$.
- $D_{12}^+ \oplus_S D_{12}^+ =$ Niemeier lattice of root system D_{12}^2.
- $O_{23} \oplus_S \mathbf{Z} = \Lambda_{24}$ the Leech lattice.

These results give added importance to Theorems 1 and 2, since the theta series of such a lattice is easy to compute if one knows the theta series of the four cosets of L_0 into L_0^* and of the four cosets of L_0' into $L_{0'}^*$. Specifically, if L and L' denote two Type I unimodular lattices of respective dimensions n and n', then the theta series of their shadow sum is

$$\theta_{L \oplus_S L'} = \sum_{i=0}^{3} \theta_{L_i} \theta_{L_i'}.$$

Additionally, if C and C' denote two Type I self-dual codes of respective lengths n and n', then the *cwe* of their shadow sum is

$$\mathrm{cwe}_{C \oplus_S C'} = \sum_{i=0}^{3} \mathrm{cwe}_{C_i} \, \mathrm{cwe}_{C_i'}.$$

5. Constant vectors and shadows

In light of Theorem 1 we would like to know when a constant vector is contained in the shadow of a unimodular lattice. As an example we note that $[1] = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is not in the shadow of any unimodular lattice formed by construction A_8 from a self-dual code over $\mathbf{Z}_8$. Since if [1] were in the shadow of the lattice then there would exist a vector s in the shadow of the code such that $\Lambda(s) = [1]$, where Λ indicates the A_8 construction. Then for some integer α we have $(1/\sqrt{8})\alpha = \frac{1}{2}$ which implies that $\sqrt{2}$ is an integer, giving a contradiction.

In general we want to know when there is a constant vector in the shadow of a code over $\mathbf{Z}_{2k}$. We shall develop a general theory and apply it to this situation.

Let C be a self-dual code over $\mathbf{Z}_{2k}$. We shall give an alternate definition of a shadow and call it the generalized shadow.

Let s be any vector in $\mathbf{Z}_{2k}^n$ such that $s\in S$, $s\notin C$, and $2s\in C$. Define a subcode of C by

$$sC_0=\{v\mid v\in C,\ [v,s]=0\}. \tag{12}$$

The code sC_0 is a subcode of index 2 in C and let $sC_2=C-sC_0$. Then $sC_0^{\perp}=C\cup sS=C\cup sC_1\cup sC_3$.

Notice that if $L=\Lambda(C)$ is the lattice formed from C then $\Lambda(sC_0)=\Lambda(s)L_0$ and $\Lambda(sS)=\Lambda(s)L_1\cup\Lambda(s)L_3$. Specifically the s-shadow is mapped via the construction to the corresponding $\Lambda(s)$ shadow of the lattice, i.e. $sL_0=\{v\mid v\cdot\Lambda(s)\in\mathbf{Z},\ \mathbf{v}\in\mathbf{L}\}$, $sL_2=sL-sL_0$, and $sS=sL_0^{\perp}-sL$. If the vector $s\in S$ where S is the standard shadow then $sC_0=C_0$ and $sS=S$.

Let η be a $4k$-th root of unity, i.e. $\eta=\exp(2\pi\mathrm{i}/4k)$. First we compute the complete weight enumerator of the standard subcode C_0.

$$\begin{aligned}\mathrm{cwe}_{C_0}(x_0,x_1,\dots,x_{2k-1})=\tfrac12(&\mathrm{cwe}_C(x_0,x_1,\dots,x_{2k-1})\\&+\mathrm{cwe}_C(x_0,\eta^{1^2}x_1,\dots,\eta^{(2k-1)^2}x_{2k-1})).\end{aligned}$$

Specifically, the second summand replaces x_i with $\eta^{i^2}x_i$.

Let s be the constant vector $s=(\alpha,\alpha,\dots,\alpha)$. Let $\mu=\exp(2\pi\mathrm{i}/2k)$. Now we can compute cwe_{sC_0} for this vector s,

$$\begin{aligned}\mathrm{cwe}_{C_0}(x_0,x_1,\dots,x_{2k-1})=\tfrac12(&\mathrm{cwe}_C(x_0,x_1,\dots,x_{2k-1})\\&+\mathrm{cwe}_C(x_0,\mu^{1\alpha}x_1,\dots,\mu^{(2k-1)\alpha}x_{2k-1})).\end{aligned}$$

Specifically, the second summand replaces x_i with $\mu^{i\alpha}x_i$.

Moreover, note that for a given monomial $x_0^{a_0}x_1^{a_1}\dots x_{2k-1}^{a_{2k-1}}$ representing a vector v we have $[v,s]=0$ if and only if

$$x_0^{a_0}x_1^{a_1}\dots x_{2k-1}^{a_{2k-1}}=x_0^{a_0}(\mu^{\alpha}x_1)^{a_1}\dots(\mu^{(2k-1)\alpha}x_{2k-1})^{a_{2k-1}}.$$

Hence, if this is a weight enumerator for a subcode D_0 then $D_0=sC_0$.

If S contains some constant vector $s=(\alpha,\alpha,\dots,\alpha)$ then $\mathrm{cwe}_{C_0}=\mathrm{cwe}_{sC_0}$ and therefore

$$\mathrm{cwe}_C(x_0,\eta^{1^2}x_1,\dots,\eta^{(2k-1)^2}x_{2k-1})=\mathrm{cwe}_C(x_0,\mu^{1\alpha}x_1,\dots,\mu^{(2k-1)\alpha}x_{2k-1}). \tag{13}$$

Theorem 5. *A shadow of a self-dual code C over $\mathbf{Z}_{2k}$ has a constant vector in the shadow S if and only if Eq.* (13) *holds for some α.*

Example. Let C be the self-dual code in $\mathbf{Z}_4^2$, $C=\{00,02,20,22\}$. With respect to the above $k=1$ and in Eq. (13) we have $\eta=\exp(2\pi\mathrm{i}/8)$ and $\mu=i$. Then $\mathrm{cwe}_C(x_0,x_1,x_2,x_3)=x_0^2+2x_0x_2+x_2^2$, and

$$\begin{aligned}\mathrm{cwe}_C(x_0,\eta^{1^2}x_1,\dots,\eta^{(2k-1)^2}x_{2k-1})&=x_0^2-x_0x_2+x_2^2\\&=\mathrm{cwe}_C(x_0,\mu^{1\alpha}x_1,\dots,\mu^{(2k-1)\alpha}x_{2k-1}).\end{aligned}$$

Hence we see that the shadow contains the all-one vector.

Let $s=(\alpha,\alpha,\ldots,\alpha)$, we can compute $\mathrm{cwe}_{sS}(x_0,\ldots,x_{2k-1})$ easily since $sS=(s+C)$, hence if $v\in C, v=(v_1,\ldots,v_n)$ then $s+v=(\alpha+v_1,\ldots,\alpha+v_n)$. This gives

$$\mathrm{cwe}_{sS}(x_0,\ldots,x_{2k-1})=\mathrm{cwe}_C(x_\alpha,x_{1+\alpha},\ldots,x_{2k-1+\alpha}). \tag{14}$$

Moreover, given that

$$\mathrm{cwe}_{sC_2}(x_0,\ldots,x_{2k-1})=\mathrm{cwe}_C(x_0,\ldots,x_{2k-1})-\mathrm{cwe}_{sC_0}(x_0,\ldots,x_{2k-1})$$

we have

$$\mathrm{cwe}_{sC_1}(x_0,\ldots,x_{2k-1})=\mathrm{cwe}_{sC_0}(x_\alpha,x_{1+\alpha},\ldots,x_{2k-1+\alpha}) \tag{15}$$

and

$$\mathrm{cwe}_{sC_3}(x_0,\ldots,x_{2k-1})=\mathrm{cwe}_{sC_2}(x_\alpha,x_{1+\alpha},\ldots,x_{2k-1+\alpha}). \tag{16}$$

So if the complete weight enumerator of C is known then it is easy to compute the complete weight enumerators of cwe_{sC_0}, cwe_{sC_2}, cwe_{sC_1}, cwe_{sC_3}, and cwe_{sS}. Moreover, the theta series of the corresponding lattices can also be computed.

Given $s=(\alpha,\alpha,\ldots,\alpha)$, a corresponding vector in the induced lattice is $1/\sqrt{2k}(\alpha,\alpha,\ldots,\alpha)$ is in the s-shadow of the lattice. Hence, it will be interesting to know when there exists a constant vector S such that $s+s\in C$ for a self-dual code C over $\mathbf{Z}_{2k}$.

Theorem 6. *Let C be a self-dual code over $\mathbf{Z}_{2k}$ then $(k,k,\ldots,k)\in C$.*

Proof. If $x\in Z_{2k}$ then $xk=0$ if $x\equiv 0\,(\mathrm{mod}\,2)$ and $xk=k$ if $x\equiv 1\,(\mathrm{mod}\,2)$.

Let $v\in C$, we have $[v,v]=0$. If $v_i\equiv 0\,(\mathrm{mod}\,2)$ then $v_i^2\equiv 0\,(\mathrm{mod}\,2)$ and if $v_i\equiv 1\,(\mathrm{mod}\,2)$ then $v_i^2\equiv 1\,(\mathrm{mod}\,2)$. Hence, there are evenly many i (denote the number by $2r$) such that $v_i\equiv 1\,(\mathrm{mod}\,2)$. Therefore $[v,(k,k,\ldots,k)]=2rk=0$. $\square$

Corollary 1. *A unimodular lattice constructed from some code via construction A_{2k} contains the constant vector $(1/\sqrt{2k})(k,k,\ldots,k)$.*

An important example of the previous corollary is that any lattice constructed from a self-dual code over $\mathbf{Z}_4$ contains the all-one vector.

Theorem 7. *If C is a self-dual code over $\mathbf{Z}_{2^r}$ of length $n\not\equiv 0\,(\mathrm{mod}\,2^r)$ then there exists a constant vector s, such that $s\notin C$ but $s+s\in C$.*

Proof. Theorem 6 gives that $(2^{r-1},2^{r-1},\ldots,2^{r-1})\in C$. There exists α such that

$$(2^\alpha,2^\alpha,\ldots,2^\alpha)\notin C$$

and

$$(2^{\alpha+1},2^{\alpha+1},\ldots,2^{\alpha+1})\in C.$$

Otherwise we would have $(1,1,\ldots,1)\in C$, but

$$[(1,1,\ldots,1),(1,1,\ldots,1)]=n\not\equiv 0\,(\mathrm{mod}\,2^r).$$

Hence $s=(2^\alpha,2^\alpha,\ldots,2^\alpha)$. $\square$

If $E_n := 2\mathbf{Z}_4^n$ then $\mathrm{cwe}_{E_n} = (x_0 + x_2)^n$. Computing the left hand of Eq. (13) we have $(x_0 - x_2)^n$ and computing the right side for $\alpha = 1$ we have $(x_0 - x_2)^n$. So the all one vector is in the shadow and is not in the code, i.e. $S = sS$, where $s = (1, 1, \dots, 1)$. Then the associated lattice is in the desired situation for Theorem 1.

Over $\mathbf{Z}_{k^2}$ with k even we have the natural generalization of the E_n given where (k) generates a self-dual code of length 1 over $\mathbf{Z}_{k^2}$.

If $C_n = (k) \times (k) \times \cdots \times (k)$ then $(k, k, \dots, k) \in C_n$ but $(k/2, k/2, \dots, k/2) \notin C_n$.

The complete weight enumerator is easily determined, i.e.

$$\mathrm{cwe}_{C_n}(x_0, \dots, x_{k^2-1}) = (x_0 + x_k)^n.$$

The left-hand side of Eq. (13) gives $(x_0 - x_k)^n$ since $\eta_{2k^2}^{k^2} = -1$ and the right-hand side of Eq. (13) gives $(x_0 - x_k)^n$ since $\mu^{k(k/2)} = (\exp(2\pi\mathrm{i}/k^2))^{k^2/2} = -1$.

In general, the lattice formed under the image of this code contains the vector

$$\Lambda_{k^2}\left(\frac{k}{2}, \frac{k}{2}, \dots, \frac{k}{2}\right) = \frac{1}{\sqrt{k^2}}\left(\frac{k}{2}, \frac{k}{2}, \dots, \frac{k}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right) = [1].$$

6. Formally self-dual codes

A code C is said to be formally self-dual with respect to a weight enumerator if the weight enumerator is held invariant by the MacWilliams relations.

Theorem 8. *Let C be a Type I code over $\mathbf{Z}_{2k}$ with odd length n. The codes $D_1 = C_0 \cup C_1$ and $D_3 = C_0 \cup C_3$ are formally-self dual (with respect to the symmetric or Hamming weight enumerators) non-linear codes.*

Proof. Let $W_C(X)$ denote either the symmetric or Hamming weight enumerator. We note that $W_{C_1}(X) = W_{C_3}(X) = \frac{1}{2}W_S(X)$ since n is odd. Let $M \cdot W_C(X)$ denote the action of the variable transformation given by the MacWilliams relations. Apply the MacWilliams relations to $W_{D_1}(X)$ and the result is

$$\begin{aligned}
\frac{1}{|D_1|}(M \cdot W_{D_1}(X)) &= \frac{1}{|C|}(M \cdot W_{C_0}(X) + M \cdot \left(\frac{1}{2}\right)(W_{C_0^\perp}(X) - W_C(X))) \\
&= \frac{1}{2}W_{C_0^\perp}(X) + \frac{|C_0^\perp|}{2|C|}W_{C_0}(X) - \frac{|C|}{2|C|}W_C(X) \\
&= \tfrac{1}{2}W_{C_0^\perp}(X) + W_{C_0}(X) - \tfrac{1}{2}W_C(X) \\
&= \tfrac{1}{2}W_C(X) + \tfrac{1}{2}W_S(X) + W_{C_0}(X) - \tfrac{1}{2}W_{C_0}(X) - \tfrac{1}{2}W_{C_2}(X) \\
&= \tfrac{1}{2}W_{C_0}(X) + \tfrac{1}{2}W_{C_2}(X) + \tfrac{1}{2}W_S(X) + \tfrac{1}{2}W_{C_0}(X) - \tfrac{1}{2}W_{C_2}(X) \\
&= W_{C_0}(X) + \tfrac{1}{2}W_S(X) \\
&= W_{D_1}(X).
\end{aligned}$$

The same computation holds for D_3, since $W_{D_1}(X)=W_{D_3}(X)$. The code is non-linear since the glue group is the cyclic group of order 4. □

As a simple example we take the self-dual code of length 1. Then $D_1=\{0,1\}$, and $\mathrm{swe}_{D_1}=x_0x_1$. Note that applying the MacWilliams relations results in $x_0^1x_1^1$, but that the same is not true for the complete weight enumerator.

Let the minimum weight of C_i be denoted by d_i then this theorem is especially useful when $d_2<d_i$ for $i=0,1,3$. Then a code is produced with higher minimum weight than the self-dual code with a weight enumerator that satisfies the MacWilliams relations.

Corollary 2. *Let C be a Type I code of odd length, with D_1 and D_3 as defined above, then $A_{2k}(D_1)$ and $A_{2k}(D_3)$ are sphere packings whose theta series are held invariant by the Poisson formula, that is*

$$\Theta_L(z)=(\det L)^{1/2}\left(\frac{\mathrm{i}}{z}\right)^{n/2}\Theta_L\left(\frac{-1}{z}\right)$$

and whose minimum norm is $\min\{2k,d_{\mathrm{E}}(D_i)\}$ *where* $d_{\mathrm{E}}(D_1)$ *is the minimum Euclidean weight of D_i, for $i=1,2$.*

We computed the swe of FSD codes obtained from cyclic self-dual $\mathbf{Z}_4$ codes of [11]. Some have a better minimum weight than the self-dual codes of the same length [12, Table XVI, p. 279]. This is the case for lengths 7, 15, 23 and 47. Based on the following data and polarization computations akin to [2], we conjecture that the codewords of fixed Lee composition support t-designs with

- $t=2$ for lengths $7,15,21,31,47$.
- $t=3$ for length 23.

Borrowing the notations of [11], we give the parameters of our formally self-dual codes in lengths $7,15,21,23,31,35$ and 47. Until length 23, we use a "$*$" when the parameter is better than any one known for this length.

- Length 7

  From the only non-trivial cyclic self-dual code $C(7,4^32,4)$, we construct a formally self-dual code with $d_{\mathrm{H}}=4*$, $d_{\mathrm{Lee}}=5*$, $d_{\mathrm{E}}=7*$ and

$$\mathrm{swe}:=a^7+7a^3c^4+42a^2b^4c+14c^3b^4+28a^3cb^3+28ac^3b^3+8b^7.$$

- Length 15

  From the only non-trivial cyclic self-dual code $C(15,4^42^7,6)$, we construct a formally self-dual code with $d_{\mathrm{H}}=4*$, $d_{\mathrm{Lee}}=7*$, $d_{\mathrm{E}}=7$ and

$$\begin{aligned}\mathrm{swe}:=\;&105a^{11}c^4+280a^9c^6+435a^7c^8+168a^5c^{10}+35a^3c^{12}+3360c^6b^7a^2\\&+a^{15}+5040b^8a^5c^2+8400b^8a^3c^4+1680b^8ac^6+3360a^6b^7c^2\\&+8400a^4b^7c^4+120a^8b^7+120c^8b^7+1024b^{15}+240b^8a^7.\end{aligned}$$

- Length 21

  There are four inequivalent non-trivial cyclic self-dual codes: $C_1 := C_{21,1}(21, 4^6 2^9, 6)$, $C_2 := C_{21,2}(21, 4^3 2^{15}, 4)$, $C_3 := C_{21,3}(21, 4^9 2^3, 4)$ and an other one C_4 generated by $(fh, 2fg)$ with $f := f_1 f_2^*$, $h := x^3 - 1$ and $fgh = x^{21} - 1$ with the notation of [11]. We obtain:

Code	d_E	d_{Lee}	d_H
C_1	8	8	4
C_2	8	4	2
C_3	8	6	4
C_4	5	5	5

- Length 23

  From the only non-trivial cyclic self-dual code $C(23, 4^{11}2, 10)$, we construct a formally self-dual code with $d_H = 8*$, $d_{Lee} = 11*$, $d_E = 15*$ and

$$
\begin{aligned}
\mathrm{swe} := {} & a^{23} + 8096b^{16}a^7 + 506a^{15}c^8 + 1288a^{11}c^{12} + 253a^7c^{16} \\
& + 127512a^{10}b^7c^6 + 2024c^{14}b^7a^2 + 8096b^{15}a^8 + 2576b^{12}c^{11} \\
& + 8096b^{15}c^8 + 202400a^8b^7c^8 + 226688b^{15}a^6c^2 \\
& + 28336a^4c^{12}b^7 + 1020096b^{11}a^7c^5 + 170016b^{16}a^5c^2 \\
& + 566720b^{15}a^4c^4 + 15456b^{11}a^{11}c + 1020096b^{11}a^5c^7 + 15456b^{11}c^{11}a \\
& + 56672b^{16}ac^6 + 127512c^{10}b^7a^6 + 283360b^{16}a^3c^4 + 28336b^{12}a^{10}c \\
& + 226688b^{15}c^6a^2 + 2024a^{14}b^7c^2 + 425040b^{12}a^8c^3 + 850080b^{12}a^4c^7 \\
& + 1190112b^{12}a^6c^5 + 318780b^8a^9c^6 + 85008b^8a^{11}c^4 + 7084b^8a^{13}c^2 \\
& + 141680b^{12}a^2c^9 + 28336a^{12}c^4b^7 + 404800b^8a^7c^8 + 28336b^8a^3c^{12} \\
& + 191268b^8a^5c^{10} + 283360b^{11}c^9a^3 + 283360b^{11}a^9c^3 \\
& + 2048b^{23} + 1012b^8ac^{14}.
\end{aligned}
$$

- Length 31

  There are five inequivalent non-trivial cyclic self-dual codes: $C_1 := C_{31,1}(31, 4^5 2^{21}, 6)$, $C_2 := C_{31,2}(31, 4^{10} 2^{11}, 10)$, $C_3 := C_{31,3}(31, 4^{10} 2^{11}, 10)$, $C_4 := C_{31,4}(31, 4^{15}2, 12)$ and $C_5 := C_{31,5}(31, 4^{15}2, 12)$ with the notation of [11]. The codes C_2 and C_3 have the same symmetric weight enumerator as do C_4 and C_5. We obtain:

Code	d_E	d_{Lee}	d_H
C_1	15	8	4
C_2, C_3	15	12	6
C_4, C_5	15	13	8

- Length 35
  There exist four inequivalent cyclic self-dual codes. We have, borrowing the notations of [11]:

Codes	Generators	d_{Lee}	d_{E}	d_{H}	t-design
1	$f_3f_{12}h_0, 2f_3f_{12}f_3^*f_{12}^*$	4	4	3	$t=1$
2	$f_3^*f_{12}h_0, 2f_3^*f_{12}f_3f_{12}^*$	8	8	6	$t=1$
3	$f_3^*f_3h_0f_{12}, 2f_{12}f_{12}^*$	6	8	3	$t=1$
4	$f_3f_{12}f_{12}^*h_0, 2f_3f_3^*$	4	8	2	$t=1$

  and we obtain four formally self-dual codes with minimum weights, respectively, $d_{\text{Lee}}=6$, $d_{\text{E}}=8$, $d_{\text{H}}=4$ for the first code, $d_{\text{Lee}}=8, d_{\text{E}}=8$, $d_{\text{H}}=6$ for the second code, $d_{\text{Lee}}=8$, $d_{\text{E}}=8$, $d_{\text{H}}=4$ for the third code and $d_{\text{Lee}}=4$, $d_{\text{E}}=8$, $d_{\text{H}}=2$ for the fourth code. Their symmetric weight enumerators can be polarized at most one time. This indicates that these codes cannot contain t-design with $t>1$.
- Length 39
  There is a unique non-trivial self-dual cyclic code ($(fh, 2ff^*)$ in the notation of [11]). From this code, we construct a formally self-dual code. The symmetric weight enumerators of the two codes can be polarized at most one time. Their parameters are:

cyclic code	FSD code
$d_{\text{H}}=3$	$d_{\text{H}}=6$
$d_{\text{Lee}}=6$	$d_{\text{Lee}}=12$
$d_{\text{E}}=12$	$d_{\text{E}}=15$

- Length 47
  We construct a formally self-dual code from the quadratic residue code over Z_4) with minimum weight, respectively, $d_{\text{Lee}}=17$, $d_{\text{E}}=23$, $d_{\text{H}}=12$ and

$$
\begin{aligned}
\text{swe} := {} & 356730a^{31}c^{16} + 2330636a^{27}c^{20} + 12972a^{35}c^{12} + 4324c^{36}a^{11} \\
& + 3840840a^{23}c^{24} + 1664740c^{28}a^{19} + 178365c^{32}a^{15} + a^{47} \\
& + 1061836032a^{22}b^{23}c^{2} + 745803520a^{19}b^{27}c + 5876246816c^{21}a^{15}b^{11} \\
& + 634538352c^{25}a^{11}b^{11} + 7387648b^{24}a^{23} + 311328b^{20}c^{27} \\
& + 53271680b^{28}c^{19} + 91322880b^{32}a^{15} + 35422208b^{36}c^{11} \\
& + 1163320312b^{12}c^{25}a^{10} + 28743591096b^{12}a^{18}c^{17} \\
& + 25717949928b^{12}a^{16}c^{19} + 10654336b^{12}c^{29}a^{6} \\
& + 259440b^{12}c^{31}a^{4} + 44139392b^{12}a^{28}c^{7} + 14690617040b^{12}a^{14}c^{21} \\
& + 9354057312b^{12}a^{22}c^{13} + 20566863856b^{12}a^{20}c^{15} + 444654216b^{12}a^{26}c^{9}
\end{aligned}
$$

$$+ 95128b^{12}a^{32}c^{3} + 5287075872b^{12}c^{23}a^{12} + 1608528b^{12}a^{30}c^{5}$$
$$+ 2643909800b^{12}a^{24}c^{11} + 8648b^{12}c^{33}a^{2} + 148218072b^{12}c^{27}a^{8}$$
$$+ 1883169108480b^{24}a^{7}c^{16} + 176972672b^{24}ac^{22}$$
$$+ 10386094688256b^{24}a^{11}c^{12} + 6277489109760b^{24}a^{9}c^{14}$$
$$+ 258497022080b^{24}a^{5}c^{18} + 8788484753664b^{24}a^{13}c^{10}$$
$$+ 68005104640b^{24}a^{19}c^{4} + 775491066240b^{24}a^{17}c^{6}$$
$$+ 3766338216960b^{24}a^{15}c^{8} + 1946699392b^{24}a^{21}c^{2}$$
$$+ 13601020928b^{24}a^{3}c^{20} + 7335233600b^{16}c^{24}a^{7}$$
$$+ 393286739120b^{16}a^{19}c^{12} + 90198640b^{16}a^{27}c^{4}$$
$$+ 22005700800b^{16}a^{23}c^{8} + 123589156064b^{16}a^{21}c^{10}$$
$$+ 471246816b^{16}c^{26}a^{5} + 2042069536b^{16}a^{25}c^{6}$$
$$+ 837531264768b^{16}a^{15}c^{16} + 1037760b^{16}a^{29}c^{2} + 12885520b^{16}c^{28}a^{3}$$
$$+ 235972043472b^{16}c^{20}a^{11} + 56176889120b^{16}c^{22}a^{9}$$
$$+ 739098898560b^{16}a^{17}c^{14} + 574854698880b^{16}c^{18}a^{13} + 69184b^{16}c^{30}a$$
$$+ 8405856b^{20}a^{26}c + 101596704b^{20}c^{25}a^{2}$$
$$+ 5064465559296b^{20}a^{12}c^{15} + 86214886912b^{20}c^{21}a^{6}$$
$$+ 646822836000b^{20}c^{19}a^{8} + 258644660736b^{20}a^{20}c^{7}$$
$$+ 1365514876000b^{20}a^{18}c^{9} + 846639200b^{20}a^{24}c^{3}$$
$$+ 5843614106880b^{20}a^{14}c^{13} + 5118232320b^{20}c^{23}a^{4}$$
$$+ 23543868672b^{20}a^{22}c^{5} + 2457673355808b^{20}a^{10}c^{17}$$
$$+ 3798222458976b^{20}a^{16}c^{11} + 4026380117760b^{28}a^{8}c^{11}$$
$$+ 1012161920b^{28}a^{18}c + 4921131255040b^{28}a^{10}c^{9}$$
$$+ 9109457280b^{28}c^{17}a^{2} + 206481031680b^{28}c^{15}a^{4}$$
$$+ 2684253411840b^{28}a^{12}c^{7} + 619443095040b^{28}a^{14}c^{5}$$
$$+ 1445367221760b^{28}c^{13}a^{6} + 51620257920b^{28}a^{16}c^{3}$$
$$+ 274242608640b^{32}a^{5}c^{10} + 41551910400b^{32}a^{3}c^{12}$$
$$+ 124655731200b^{32}a^{11}c^{4} + 1369843200b^{32}ac^{14}$$

$$+457071014400b^{32}a^9c^6+587662732800b^{32}a^7c^8$$
$$+9588902400b^{32}a^{13}c^2+16365060096b^{36}c^5a^6$$
$$+389644288b^{36}a^{10}c+5844664320b^{36}a^8c^3$$
$$+11689328640b^{36}c^7a^4+1948221440b^{36}c^9a^2$$
$$+166207641600a^4b^{31}c^{12}+9076923504a^{19}c^{17}b^{11}$$
$$+9076923504a^{17}c^{19}b^{11}+3113280a^{27}b^{19}c$$
$$+98812048c^{27}b^{11}a^9+311328c^{31}a^5b^{11}$$
$$+2440188864a^{13}c^{23}b^{11}+42081583104c^5a^7b^{35}$$
$$+9132288c^{29}a^7b^{11}+17296c^{33}b^{11}a^3$$
$$+2824753662720a^8b^{23}c^{16}+637599744b^{35}c^{11}a$$
$$+637599744b^{35}a^{11}c+17296a^{33}c^3b^{11}$$
$$+2440188864a^{23}c^{13}b^{11}+3693824a^{24}b^{23}$$
$$+516994044160a^{18}b^{23}c^6+44941511296c^{10}a^{22}b^{15}$$
$$+40803062784a^{20}b^{23}c^4+25771040a^{28}b^{15}c^4$$
$$+7532986931712a^{10}b^{23}c^{14}+98812048c^9a^{27}b^{11}$$
$$+328488399360c^{18}a^{14}b^{15}+2234248505280c^{11}a^{17}b^{19}$$
$$+44941511296a^{10}c^{22}b^{15}+157314695648c^{20}a^{12}b^{15}$$
$$+276736c^{30}a^2b^{15}+42510800640a^3c^{17}b^{27}$$
$$+9132288c^7a^{29}b^{11}+7335233600c^8a^{24}b^{15}$$
$$+2824753662720c^8a^{16}b^{23}+3895742737920c^{15}a^{13}b^{19}$$
$$+157314695648a^{20}c^{12}b^{15}+634538352a^{25}c^{11}b^{11}$$
$$+7532986931712c^{10}a^{14}b^{23}+7335233600c^{24}a^8b^{15}$$
$$+11689328640a^9c^3b^{35}+10386094688256c^{12}a^{12}b^{23}$$
$$+42081583104b^{35}c^7a^5+11689328640b^{35}c^9a^3$$
$$+25771040c^{28}a^4b^{15}+628329088c^{26}a^6b^{15}$$
$$+123164124160c^{21}a^7b^{19}+718692040000c^{19}a^9b^{19}$$
$$+328488399360a^{18}b^{15}c^{14}+10236464640c^{23}a^5b^{19}$$

$$+516994044160c^{18}b^{23}a^6+5876246816a^{21}c^{15}b^{11}$$
$$+311328a^{31}c^5b^{11}+628329088a^{26}c^6b^{15}$$
$$+3895742737920c^{13}a^{15}b^{19}+40803062784c^{20}b^{23}a^4$$
$$+418765632384a^{16}c^{16}b^{15}+3113280c^{27}b^{19}a$$
$$+1061836032c^{22}b^{23}a^2+338655680c^{25}b^{19}a^3$$
$$+3693824c^{24}b^{23}+2234248505280a^{11}b^{19}c^{17}$$
$$+123164124160a^{21}b^{19}c^7+718692040000a^{19}c^9b^{19}$$
$$+8388608b^{47}+10236464640a^{23}b^{19}c^5$$
$$+338655680a^{25}b^{19}c^3+276736a^{30}c^2b^{15}$$
$$+2890734443520a^7c^{13}b^{27}+2890734443520c^7b^{27}a^{13}$$
$$+745803520c^{19}b^{27}a+42510800640a^{17}b^{27}c^3$$
$$+578146888704a^5b^{27}c^{15}+6263257960960c^9b^{27}a^{11}$$
$$+6263257960960c^{11}b^{27}a^9+578146888704a^{15}b^{27}c^5$$
$$+731313623040a^6c^{10}b^{31}+10958745600a^2c^{14}b^{31}$$
$$+166207641600c^4b^{31}a^{12}+91322880a^{16}b^{31}$$
$$+91322880c^{16}b^{31}+10958745600a^{14}b^{31}c^2$$
$$+731313623040c^6b^{31}a^{10}+1175325465600c^8b^{31}a^8.$$

References

[1] E. Bannai, S.T. Dougherty, M. Harada, M. Oura, Type II codes, even unimodular lattices, and invariant rings, IEEE Trans. Inform. Theory 45 (4) (1999) 1194–1205.

[2] A. Bonnecaze, E.M. Rains, P. Solé, 3-Colored 5-Designs and $\mathbf{Z}_4$-codes, J. Statist. Plann. Inference 86 (2) (2000) 349–368.

[3] Y. Choie, S.T. Dougherty, H. Kim, Complete joint weight enumerators and self-dual codes, to appear in IEEE Trans. Inform. Theory.

[4] Y. Choie, N.-S. Kim, The complete weight enumerator of a Type II code over $\mathbf{Z}_{2m}$ and Jacobi forms, IEEE Trans. Inform. Theory 47 (2001) 396–399.

[5] J.H. Conway, N.J.A. Sloane, A new upper bound on the minimum distance of self-dual codes, IEEE Trans. Inform. Theory 36 (1990) 1319–1333.

[6] J.H. Conway, N.J.A. Sloane, A new upper bound for the minimum of an integral lattice of determinant 1, Bull. AMS 23 (2) (1990) 383–387.

[7] J.H. Conway, N.J.A. Sloane, Sphere Packings, Lattices and Groups, 3rd Edition, Springer, Berlin, 1998.

[8] S.T. Dougherty, M. Harada, P. Solé, Shadow lattices and shadow codes, Discrete Math. 219 (2000) 49–64.

[9] S.T. Dougherty, P. Solé, Shadow of codes and lattices, submitted to Proceedings to Third Asian Math Conference, 2000.
[10] M. Eichler, D. Zagier, The Theory of Jacobi forms, Birkhäuser, Basel, 1985.
[11] V. Pless, P. Solé, Z. Qian, Cyclic self-dual Z_4-codes, Finite Fields Appl. 3 (1997) 48–69.
[12] E.M. Rains, N.J.A. Sloane, in: V. Pless, W.C. Huffman (Eds.), Handbook of Coding Theory, North-Holland, Elsevier, Amsterdam, 1998.